BURIED POINTS IN JULIA SETS

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Abstract. An introduction to buried points in Julia sets and a list of questions about buried points, written to encourage aficionados of topology and dynamics to work on these questions.

Dedicated to Bob Devaney on the occasion of his 60th birthday.

1. Introduction

Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational function, where $\mathbb{C}_\infty$ denotes the Riemann sphere. The *Fatou set* of R, denoted $F(R)$, is the domain of normality for the family of functions $\{R^i \mid i \in \mathbb{N}\}$. A component of the Fatou set is called a *Fatou component*. The *Julia set* of R, denoted $J(R)$, is the complement of $F(R)$. General references for Juia sets of rational functions are [Bea91], [Mil06], and [CG93]; we present a few facts below.

In the case that the degree of R is at least two, the Julia set is a non-empty, compact, perfect subset of $\mathbb{C}_\infty$. It is either nowhere dense in $\mathbb{C}_\infty$ or equal to $\mathbb{C}_\infty$. (We are more interested in the former case, and assume it is so henceforth.) It is well-known that $J(R^n) = J(R)$ and $F(R^n) = F(R)$ for any integer $n \geq 1$. The Julia set and Fatou set are each *fully invariant* under R, meaning that $R^{-1}(J(R)) = J(R)$ and $R^{-1}(F(R)) = F(R)$. The restriction $R|_{J(R)}$ is *topologically exact*: if $U \subset J(R)$ is open in $J(R)$, there exists $n \in \mathbb{N}$ such that $R^n(U) = J(R)$.

The notion of a *buried point* is purely topological, but we consider it here just in the case of Julia sets in $\mathbb{C}_\infty$.

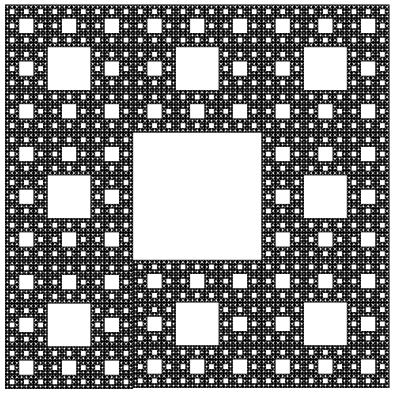
Definition 1 (Buried Points)**.** A point of a Julia set $J(R)$ is said to be *buried* if it does not belong to the boundary of a Fatou component. The set of all buried points of $J(R)$ is called the *residual Julia set*, denoted $J'(R)$.

Date: October 23, 2008.

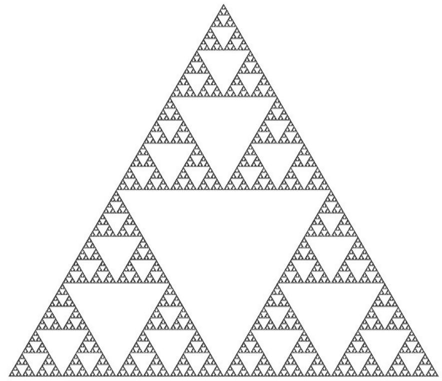
2000 *Mathematics Subject Classification.* Primary: 37F20; Secondary: 54F15.

Key words and phrases. Julia set, holomorphic dynamics, Fatou component, complex dynamics, buried point, residual Julia set.

We thank Murat Tuncali for inspiration.



(a) The Sierpinski carpet.



(b) The Sierpsinski gasket.

Polynomial Julia sets have no buried points, because the Fatou component containing ∞ has the Julia set as its boundary. Curt McMullen [McM88] presented the first examples of rational maps with non-empty residual Julia sets. He showed that functions of the form $z \mapsto z^n + \lambda/z^d$ have Julia sets homeomorphic to the product of a Cantor set and a circle whenever $1/n + 1/d < 1$ and λ is sufficiently small. In this case, the Julia set is not connected and there are uncountably many components of the Julia set which do not intersect the boundary of any Fatou component. Later, John Milnor and Tan Lei [ML93] and Bob Devaney, Dan Look, and David Uminsky [DLU05] exhibited rational functions with Julia sets homeomorphic to the Sierpinski carpet (see Figure 1(a)).

Since $R|_{J(R)}$ is topologically exact, we have that the residual Julia set is non-empty if and only if the boundary of each Fatou component is nowhere dense in $J(R)$. As a consequence, the residual Julia set is a dense G_δ subset of $J(R)$ whenever it is non-empty, as there are only countably many Fatou components. It is also nowhere locally compact (because the union of the boundaries of Fatou components is also dense). The residual Julia set and the union of boundaries of Fatou components are each fully invariant subsets of $J(R)$.

Interestingly, the known examples of Julia sets which have non-empty residual Julia sets are precisely the examples for which no Fatou component has a finite grand orbit. Peter M. Makienko made the following conjecture.

Makienko's Conjecture. Let $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational function. The Julia set $J(R)$ has buried points if and only if there is no completely invariant component of the Fatou set of R^2.

This conjecture was formulated as a possible entry in Sullivan's dictionary of correspondences between Kleinian group actions and iteration theory of rational functions in 1990 [EL90] as a parallel to a theorem of Abikoff. Note that one direction is easily proved with facts already given. Specifically, if a Fatou component F is completely invariant under R^2, then $\partial F \subset J(R)$ is also completely invariant under R^2 and closed, so $\partial F = J(R^2) = J(R)$ [Mil06, Corollary 4.13]. The example $z \mapsto \frac{1}{z^2}$ illustrates why one must examine the Fatou set of R^2. However, it is known [Bea91, Theorem 9.4.3] that a rational map may have at most two completely invariant Fatou components, in which case the Julia set is a simple closed curve.

Makienko's conjecture has received attention in the past, with results being limited by topological considerations: see Morosawa [Mor97, Mor00], Qiao [Qia97], and Sun and Yang [SY03]. Clinton Curry, John Mayer, Jonathan Meddaugh, and Jim Rogers [CMMR08] recently proved the following.

Theorem (Makienko's Conjecture Holds for Decomposable Julia Sets of Rational Maps). *If R is a rational function such that $J(R)$ has no buried points and $F(R^2)$ has no completely invariant components, then $J(R)$ is an indecomposable continuum.*

Recall that a continuum is *decomposable* if it can be written as the union of two proper subcontinua; otherwise it is *indecomposable*. There are no known examples of Julia sets which are indecomposable continua. In fact, whether or not there exists a rational function with an indecomposable continuum as its Julia set is a well-known unsolved problem [MR93].

2. Questions

While Makienko's Conjecture is probably the primary motivation for dynamicists to consider questions about buried points in Julia sets, there are a host of topological questions that one could consider that seem interesting from a topological viewpoint. We present some of them here.

We remarked earlier that the Sierpinski carpet (the Sierpinski universal plane curve [Nad92, 1.11], see Figure 1(a)) appears as a Julia set for some rational functions. Sierpinski carpet Julia sets are ubiquitous in the families $z \to z^n + \frac{\lambda}{z^d}$ for $n \geq 1,\ d \geq 2$ studied by Devaney, Paul Blanchard, and their students [BlaDev06]. The set of buried points of the Sierpinski carpet is the set of so-called *irrational points* of the Sierpinski carpet, studied by Krasinkiewicz [Kra69]. They are one of the two orbits in the Sierpinski carpet under its group of homeomorphisms,

the points of boundaries of complementary components forming the other orbit. This set of buried points is one-dimensional, connected, and locally path connected, and it contains a copy of every planar curve (one-dimensional plane continuum). The latter is proved using Whyburn's characterization of the Sierpinski carpet [W58].

The Sierpinski gasket (triangular Sierpinski curve, see Figure 1(b)) and generalizations of it also appear as Julia sets in the aforementioned families as studied by Devaney and Monica Morena-Rocha [DRS07]. In this case, the set of buried points is 0-dimensional, and in fact can be shown to be homeomorphic to the irrational points on the real line.

As previously noted [McM88], McMullen showed that the set of buried points of a Julia set could be the "irrational" factors of a Cantor set cross a circle. Such Julia sets occur in all the families $z \to z^n + \frac{1}{z^n}$ for $n \geq 3$ and small λ [D05].

In [BDGMR, BDGR], Blanchard, Devaney, Antonio Gajiro, Sebastian Marotta, and Elizabeth Russell study the family $z \to z^n + c + \frac{\lambda}{z^n}$ and observe some more complicated residual Julia sets. However, the residual Julia sets in their examples appear to decompose into mutually dense subsets homeomorphic to one of the "irrational" spaces previously noted.

Definition 2. A space X is *homogeneous* iff there is exactly one orbit in its group of homeomorphisms. That is, given $x, y \in X$, there is a homeomorphism $h : X \to X$ with $h(x) = y$. A space X is $1/n$*-homogeneous* iff there are exactly n orbits in its group of homeomorphisms, each of which is dense.[1]

Note that homogeneous is 1/1-homogeneous. The Sierpinski carpet is an example of a 1/2-homogeneous space. In each of the aforementioned examples of residual Julia sets, the set of buried points is either a homogeneous space or a 1/2-homogeneous space. That is not to say that the homeomorphisms can necessarily be extended to the Julia set.

Question 1. Is there a residual Julia set whose components are not either (irrational) points, (irrational) circles, or the irrational points of the Sierpinski curve.

Question 2. Is the residual Julia set always $1/n$-homogeneous for some n? Can the homeomorphism be extended to the Julia set?

The reader can readily verify that the answer to the second part of Question 2 is "no" in the case of the Sierpinski gaskets, and "yes" when

[1]The requirement that each orbit be dense in X is not part of the standard definition.

the Julia set is itself the Sierpinski carpet. One can define both stronger and weaker homogeneity properties, which have received attention in topology, and formulate questions similar to Question 2.

As noted earlier, Krasinkiewicz showed that in the Sierpinski carpet any buried point can be carried by a homeomorphism of the carpet to any other buried point [Kra69]. In [CMR06] it is proved that any buried arc (homeomorphic image of $[0,1]$) can be mapped to any other buried arc by a homeomorphism of the carpet. This motivates the following question.

Question 3. Let $S \subset \mathbb{C}_\infty$ be a Sierpinski carpet. Suppose that X and Y are continua in the buried points of S that are equivalently embedded in $\mathbb{C}_\infty$ (i.e., there exists a homeomorphism $h : \mathbb{C}_\infty \to \mathbb{C}_\infty$ such that $h(X) = Y$). Is there a homeomorphism $g : S \to S$ such that $g(X) = Y$?

The assumption that X and Y be equivalently embedded in $\mathbb{C}_\infty$ is necessary, since any homeomorphism from S to itself can be extended to a homeomorphism of $\mathbb{C}_\infty$ [Kra69].

Question 4. What does local connectivity imply about the residual Julia set? For example, is a Julia set with connected buried point set locally connected iff the set of buried points is locally connected?

For Julia sets, local connectedness implies connectedness. This follows from the fact that $R|_{J(R)}$ is topologically exact; if $U \subset J(R)$ is connected and open, then the iterate $R^n(U)$ which equals J must also be connected. However, there are connected polynomial Julia sets which are not locally connected [Mil06, Corollary 18.6]. Recently, Pascale Roesch showed that there are "genuine" rational Julia sets that are connected but not locally connected [Roe06]. The rational map is not conjugate to a polynomial, but the non-local-connectivity is achieved through reference to polynomials. It would be interesting to determine what the residual Julia set is in Roesch's examples.

Suppose that $J(R)$ is a locally connected Julia set with totally disconnected residual Julia set $J'(R)$. We conjecture that $J'(R)$ is 0-dimensional, which would imply that it is homeomorphic to the irrational real numbers. However, we have no proof. In the case that $J(R)$ is connected, but not locally connected, the situation is even less clear.

Definition 3 (0-Dimensional, Almost 0-Dimensional)**.** A space X is 0-*dimensional* at a point $x \in X$ if there is a basis at x of open sets whose boundaries are empty. X is 1-*dimensional* at x if there is a basis at x of open sets whose boundaries are 0-dimensional. X is *almost* 0-*dimensional* at a point $x \in X$ iff X is 1-dimensional at x and x has

a basis of neighborhoods U such that $X \setminus U$ is the union of countably many sets with empty boundary.

Question 5. What can be said about the topological dimension of the set of buried points? For example, if the set of buried points is totally disconnected, is it 0-dimensional? If not 0-dimensional, is it almost 0-dimensional?

Totally disconnected compact Hausdorff spaces are always 0-dimensional. However, it is well-known that totally disconnected complete metric spaces can be any dimension. In particular, topologically complete totally disconnected subsets of $\mathbb{C}_\infty$ can be 1-dimensional and not almost 0-dimensional. Of course, such examples are nowhere locally compact.

Question 6. What can be said about the Hausdorff dimension of the set of buried points? For example, can a Julia set and its (nonempty) residual Julia set have different Hausdorff dimensions? In particular, what about a Sierpinski gasket Julia set?

Useful references for Hausdorff dimension and related topics include [Fal03] and [Fal86].

Question 7. What does the nature of the post-critical set say about the buried point set?

It is known that if the post-critical set is finite, then the Julia set is locally connected [Mil06, Theorem 19.7].

Question 8. For a connected Julia set, if a critical value lies on the boundary of a Fatou component, is the set of buried points disconnected? When is the set of buried points totally disconnected?

Sierpinski gasket Julia sets in the family $z \to z^n + \frac{\lambda}{z^d}$ arise when a critical point is on the boundary of the Fatou component B_∞ containing ∞ and the critical value is pre-periodic [DRS07, Theorem 3.1]. Since all Fatou components are pre-images of B_∞, this serves to (totally) disconnect the residual Julia set.

Question 9. What can be said about invariant or periodic continua in the set of buried points?

For example, in the case of the Sierpinski gasket Julia sets, since boundaries of Fatou components eventually map to ∂B_∞, which is invariant, nearly all of the periodic points **are** buried.

Question 10. What can be said about wandering non-degenerate continua in the set of buried points?

Blokh and Levin [Blo02] bound the number and valence of wandering continua for polynomials.

Question 11. Assuming the existence of an indecomposable rational Julia set with buried points, what can be said about its residual Julia set?

Given a point x in the continuum X, the *composant* of x is the union of all proper subcontinua of X containing x. Composants are dense [Nad92, 5.20] in X, and for an indecomposable continuum are pairwise disjoint [Nad92, Theorem 11.17] and uncountable in number [Nad92, Theorem 11.15]. There is a relationship between buried points and *internal composants* of an indecomposable plane continuuum studied by Krasinkiewicz [Kra72]. For example, the boundary of a Fatou component is always in an *accessible*, hence *external* composant. If the residual Julia set is non-empty, then all internal composants are in the residual Julia set. Because composants are dense, it seems that part of each external composant must also be in the residual Julia set, unless the union of the boundaries of infinitely many Fatou components is connected.

3. Extensions

In view of Theorem 1, any counterexample to Makienko's conjecture must be exceedingly complicated. In fact, it is not known if the Julia set of a rational function can be as complicated as required. We restate here the questions which appear in [CMMR08, CMR06, CMTT06, SY03, R98, MR93].

Question 12. Can the Julia set of a rational function be an indecomposable continuum?

Question 13. Can the Julia set of a rational function contain an indecomposable subcontinuum with interior?

Theorems in [CMR06, CMTT06] indicate that the answers to these two questions are the same for polynomial Julia sets and for rational functions whose Julia set contains no buried points.

Several authors have extended the study of Makienko's conjecture to transcendental entire and meromorphic functions. Domínguez and Fagella [DF07] survey the current state of affairs in this direction. Also see Ng, Zheng, and Choi [NZC06], who prove Makienko's conjecture for locally connected Julia sets of certain meromorphic functions. The techniques of proof of Theorem 1 fail dramatically for trancendental functions. A particular point of difficulty is summarized in the following question.

Question 14. Let f be a transcendental entire or meromorphic function. If V is a Fatou component such that ∂V is nowhere dense in $J(f)$, does it follow that $\partial f(V)$ is also nowhere dense in $J(f)$?

It is also worth noting that the proof of Theorem 1 makes use of the non-existence of wandering Fatou components for rational functions proved by Dennis Sullivan [Bea91, Theorem 8.1.2], which in general is not true for transcendental functions [Berg93].

Variations of most of the questions in Section 2 could be formulated for trancendental functions. Julia sets of the family $z \to \lambda e^z$ would be a place to start.

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